Outline. Geometric estimates under curvature bounds.

1. Elements from mathematical general relativity
2. Injectivity radius of an observer
3. Local canonical foliations of an observer
4. Geometric estimates in past canonical neighborhoods

http://philippelefloch.wordpress.com
1. Elements from mathematical general relativity

\((M, g)\): time-oriented Lorentzian \((n + 1)\)-manifold

- Lorentzian metric: \(X \in T_p M\) is time-like, null, or space-like if \(g(X, X)\) is negative, zero, or positive.

- Time-like curve (observer), null curve (photon).

- Time-orientation: distinguish between future and past directions.
Causality.

- **Chronological future** $\mathcal{I}^+(p)$
  Set of points $q \in M$ attainable from $p$ by a future-oriented, time-like continuous curve (a trip) $\gamma : (a, b) \to M$.

- **Causal future** $\mathcal{J}^+(p)$

- **Future domain of dependence** $\mathcal{D}^+(S)$
  Set of points $p \in M$ such that every past-endless trip containing $p$ meets the given set $S$.

- **Future Cauchy hypersurface in $M$** $\mathcal{D}^+(S) = M$

Think of imposing initial data on $S$. 
Matter spacetime.

$(M, g)$ satisfying Einstein’s field equations

\[ G = \kappa \, T, \quad G_{\alpha\beta} = \kappa \, T_{\alpha\beta}. \]

- Einstein tensor
- Ricci tensor
- Scalar curvature

In coordinates, the $G_{\alpha\beta}$’s contain (at most) second-order derivatives of the $g_{\alpha\beta}$’s.
The energy-momentum tensor satisfies the (contracted, second) Bianchi identity ($\delta$: formal adjoint of $d$)

$$\delta T = 0,$$

that is,

$$\nabla_\beta T^{\alpha\beta} = 0$$

with, for a perfect fluid,

$$T = (\mu + p)u \otimes u + pg$$

- Unit time-like velocity vector $u$
- Proper mass-energy density $\mu \geq 0$
- Pressure $p = p(\mu)$, satisfying the dominant energy condition: for every time-like vector $X$, the time-like energy flux $T(X, \cdot)$ is causal, with $T(X, X) \geq 0$.

Euler equations of fluid dynamics.
Formulation of the initial value problem.

Initial data set.

- Riemannian 3-manifold \((\overline{M}, \overline{g})\)
- Symmetric 2-covariant tensor field [2nd fund. form] \(\overline{k}\)
- Matter fields (energy density, current) [measured orthog.] \(\overline{\rho}, \overline{J}\)
- Einstein’s constraint equations [Gauss-Codazzi eq.]

\[
\bar{R} + (\text{tr} \bar{k})^2 - |\bar{k}|^2 = 16\pi \bar{\rho}, \quad \delta \bar{k} - d(\text{tr} \bar{k}) = 8\pi \bar{J}^\flat.
\]

These are nonlinear elliptic equations.
**Initial value problem.** Future globally hyperbolic development of the initial data set

- Lorentzian manifold satisfying Einstein equations \((M, g)\)
- Foliation with normal 1-form \(n\)
- Embedding \(\psi: \bar{M} \to \mathcal{H}_0 \subset M\)
- Induced metric \(\bar{g}\)
- Second fundamental form \(\bar{k}\)
- Matter fields \(\rho, J\)

\[
\rho := T(n, n) = T^{\alpha\beta} n_\alpha n_\beta, \quad J^b := T(n, \cdot) + T(n, n) n,
\]

\(\bar{\rho}, \bar{J}\) being their restrictions to \(\mathcal{H}_0\) (\(\bar{J}\) being tangent).
Huge literature on solving the Einstein’s field equations.

- Einstein constraint equations
  Bartnik, Choquet-Bruhat, Isenberg, etc.

- Vacuum or matter models / large data / small times:
  *Maximal globally hyperbolic* development of the initial data set.
  Choquet-Bruhat, Geroch, etc

- Vacuum / small data / geodesically complete
  Stability of the Minkowski space established by Christodoulou and Klainerman. Improved by Bieri.
  Friedrich, Andersson-Moncrief, Klainerman-Rodnianski,
  Choquet-Bruhat–Moncrief, Lindblad–Rodnianski.
- Models with symmetries / large data / Penrose conjecture
  - Vacuum Moncrief, Isenberg, etc
  - Scalar field Christodoulou, Dafermos, etc
  - Vlasov’s kinetic model Andreasson, Rendall, etc
  - Euler equations of compressible fluids (weak solutions, shock / gravitational waves) PLF, Rendall, Stewart

Issues raised by physics:

- Boundary of the maximal hyperbolic development
- Nature of “singularities” (coordinate or geometric)
- Penrose’s strong censorship conjecture (generic inextendibility)

**NEED:** optimal control of the geometry in terms of the curvature
2. Injectivity radius of an observer


**Riemannian manifolds.**

$(M, g)$: complete Riemannian $n$-manifold; $B_g(p, r)$: geodesic ball centered at $p \in M$

$$\|Rm_g\|_{L^\infty(B_g(p,1))} \leq K_0.$$  

Cheeger, Gromov, and Taylor: injectivity radius $\geq c_0(K_0, n) \text{Vol}_g(B_g(p, 1)) =: i_0$

$$\exp_p : B_{gp}(0, i_0) \subset T_p M \rightarrow B_g(p, i_0) \subset M.$$  

Jost and Karcher: given $\varepsilon > 0$, one can cover $B_g(p, i_0)$ by a chart of harmonic coordinates (best regularity)

$$e^{-\varepsilon} g_E \leq g \leq e^{\varepsilon} g_E,$$

$$\|g\|_{W^{2,a}(B_g(p,i_0))} \leq C_{\varepsilon,a}, \quad a \in [1, \infty).$$

$g_E$ : Euclidian metric
**Objective.** Lower bound on the Lorentzian injectivity radius.

- As in the Riemannian case, seek for a purely local and geometric estimate “a la Cheeger, Gromov, and Taylor”.

- Do not prescribe a priori a foliation or a coordinate chart.

**Remark.** In the next section, a Lorentzian analogue of Jost and Karcher’s theorem.
Related results on Einstein spacetimes.

- Anderson

\[ \|Rm_g\|_{L^\infty} \leq K_0 \]

(plus other foliation conditions)

- Program on the long-time evolution for the Einstein equations
- *Local coordinates based on a geodesic time foliation and spatially harmonic coordinates*

- Klainerman and Rodnianski

\[ \sup_{\mathcal{H} \text{ spacelike}} \|Rm_g\|_{L^2(\mathcal{H})} \leq K_0 \]

(plus other foliation conditions)

- Program on the Einstein equations via harmonic analysis / \(L^2\) curvature conjecture
- *Geometry of null cones in vacuum spacetimes*
- Breakdown criterion
Lorentzian notion of injectivity radius.

$(\mathcal{M}, g, p, T_p) : \text{time-oriented, pointed, Lorentzian } (n + 1)\text{-manifold}$

$(p, T_p) : \text{observer}$

$T_p : \text{future-oriented, unit time-like vector}$

- **Reference inner product at } p$
  - Orthonormal basis $e_\alpha (\alpha = 0, 1, \ldots, n)$ at $p$
    
    $e_0 := T_p,$
    
    $e_j \ (j = 1, \ldots, n)$ are spacelike

  - Using the dual basis $e^\alpha$ at $p$,
    
    \[ g = -e^0 \otimes e^0 + e^1 \otimes e^1 + \ldots + e^n \otimes e^n \]

    we define

    \[ g_T = g_{T_p,p} := e^0 \otimes e^0 + e^1 \otimes e^1 + \ldots + e^n \otimes e^n \]

  - Use $g_T$ to compute the norms $|A|_{g_T}$ of tensors at $p$

- **Reference Riemannian metric $g_T$, once a field of observers $T$ is prescribed.**
Exponential map:

\[ \exp_p : B_{g_p}(0, i_0) \subset T_pM \to B_{g}(p, i_0) \subset M. \]

- Geodesics from \( p \)
- \( \exp_p \) (only) defined in a neigh. of \( 0 \in T_pM \) (if incomplete)
- Focus on the ball \( B_{g_T,p}(0, r) \subset T_pM \) determined by \( g_{T,p} \)
- We always assume that \( r \) is restricted so that \( B_{g_T,p}(0, r) \subset \subset M \).

**Definition**

The *injectivity radius of an observer*

\[ \text{Inj}(M, g, p, T_p) \]

is the supremum of all radii \( r \) such that \( \exp_p \) is a global diffeomorphism from \( B_{g_T,p}(0, r) \) to a neigh. \( B_T(p, r) \) of \( p \in M \).
Riemann curvature bound of an observer.

- By $g$-parallel transporting $T_p$, we define a field $T_\gamma$ along any radial geodesic $\gamma : [0, r] \to M$ from $p$.

- Using the reference metric $g_{T\gamma}$, we compute

\[
\sup_{[0,r]} |Rm_g|_{g_{T\gamma}}
\]

- If two distinct radial geodesics $\gamma$ and $\gamma'$ meet, $T_\gamma$ and $T_{\gamma'}$ are generally distinct.

- Riemann curvature norm of the observer

\[
Riem_r(p, T_p) := \sup_{\gamma} \sup_{[0,r]} |Rm_g|_{g_{T\gamma}}.
\]
Investigate the geometry of the covering

\[ \exp_p : B_{g_{T_p},p}(0,r) \to B_{g_{T_p}}(p,r) \subset M, \]

assuming that \( B_{g_{T_p},p}(0,r) \subset \subset M \).

**Theorem (Lower bound on the Lorentzian injectivity radius).**

If \((M, g, p, T_p)\) is pointed Lorentzian \((n+1)\)-manifold satisfying the curvature bound

\[ \mathcal{R} \text{iem}_r(p, T_p) \leq r^{-2} \]

then, for some \( c(n) \in (0,1] \),

\[ \frac{\text{Inj}(M, g, p, T_p)}{r} \geq c(n) \frac{\text{Vol}_g(B(p, c(n) r))}{r^{n+1}}. \]
Outline of the proof.

- **Lorentz geodesic foliation.**
  - Take $q \in I^-(p)$ in the past of the observer, and introduce the time function $\tau := d_g(\cdot, q)$.
  - Geodesic foliation with normal $N := \nabla \tau$ which is regarded as a new field of observers (with $N_p = T_p$)
  - Hessian of distance functions are controlled by curvature. In particular, uniform control of the second fundamental form of the geodesic slices.
Key observation.

The reference Riemannian metric $g_N$ has bounded curvature:

$$|\text{Rm}_{g_N}|_{g_N} \leq C.$$ 

Conjugacy and injectivity radii.

Use techniques of Riemannian geometry for $g_N$ and translate the estimates back to the Lorentzian metric $g$.

- Jacobi field estimates / conjugate radius estimate
- Cheeger, Gromov, and Taylor’s estimate for the length of smallest loop at $p$ / injectivity radius estimate in terms of volume.
3. Local canonical foliation of an observer

Collaboration with B.-L. Chen (Guangzhou). 

Objective.

▶ Constant mean curvature (CMC) foliation by spacelike hypersurfaces

▶ Spatially harmonic coordinates for Einstein vacuum spacetimes

▶ Defined *locally* near the observer

▶ Quantitative estimates involving curvature and injectivity bounds, only.
Earlier works.

▶ $R^m$ bounded:
  ▶ Anderson: coordinates based on a *geodesic* foliation. Does not yield the optimal regularity.

▶ $\nabla R^m$ bounded:
  ▶ Bartnik-Simon: CMC foliations in Minkowski space
  ▶ Gerhardt: global CMC foliations in Lorentzian manifolds
  ▶ Andersson-Moncrief: CMC-harmonic coordinates
A local canonical CMC foliation. Given $\theta$ sufficiently small, a local canonical CMC foliation of the observer $(p, T_p)$ is:

- a foliation by spacelike hypersurfaces $\Sigma_t$ of constant mean curvature $t$
  \[
  \bigcup_{t \leq t \leq \bar{t}} \Sigma_t \ni p,
  \]
- for some $s \in [\theta, 2\theta]$
  \[
  t := (1 - \theta) \frac{n}{sr}, \quad \bar{t} := (1 + \theta) \frac{n}{sr}
  \]
- the unit normal $N = \nabla t$, the lapse $\lambda := \left( -g(\nabla t, \nabla t) \right)^{1/2}$ and the second fundamental form $h$ satisfy
  \[
  -g(N, T) \leq \theta^{-1}, \quad \theta \leq -r^{-2} \lambda \leq \theta^{-1}, \quad r |h| \leq \theta^{-1}
  \]
- the field $T$ being defined by parallel translating $T_p$ along radial geodesics.
Theorem (Existence of local canonical foliations).

There exist constants $c(n), \theta(n) > 0$ such that, given any pointed Lorentzian manifold $(M, g, p, T_p)$ satisfying at some scale $r > 0$

$$Riem_r(p, T_p) \leq r^{-2}, \quad \text{Inj}(M, g, p, T_p) \geq r,$$

the Riemannian ball $B_T(p, c(n)r)$ can be covered by a local canonical CMC foliation of the observer. (with $\theta = \theta(n)$).
Local CMC-harmonic coordinates of an observer.

Let $(M, g, p, T_p)$ be a pointed Einstein vacuum spacetime satisfying curvature and injectivity bounds at the scale $r > 0$.

Then, there exist local coordinates $x = (t, x^1, \ldots, x^n)$

$$|t - r_1| < c^2 r, \quad ((x^1)^2 + \ldots + (x^n)^2)^{1/2} < c^2 r,$$

$$x(p) = (r_1, 0, \ldots, 0) \quad \text{for some } r_1 \in [cr, cr]$$

(now $t$ is normalized to be of order $r$), so that the following properties hold:
Theorem (Existence of local CMC-harmonic coordinates).

- \( \Sigma_t \) (constant \( t \)) spacelike hypersurface with CMC \( c^{-1}r^{-2}t \)
- \((x^1, \ldots, x^n)\) spatially harmonic for the metric induced on \( \Sigma_t \)
- Lorentzian metric

\[
g = -\lambda(x)^2 \,(dt)^2 + g_{ij}(x)(dx^i + \xi^i(x) \,dt)(dx^j + \xi^j(x) \,dt)
\]

close to Minkowski

\[
e^{-C} \leq \lambda \leq e^C, \quad e^{-C} \delta_{ij} \leq g_{ij} \leq e^C \delta_{ij}, \quad |\xi|_g^2 := g_{ij} \xi^i \xi^j \leq e^C
\]

- For \( q \in [1, \infty) \) and \( Q(n, q) > 0 \)

\[
r^{-n+q} \int_{\Sigma_t} |\partial g|^q \,dv_{\Sigma_t} + r^{-n+2q} \int_{\Sigma_t} |\partial^2 g|^q \,dv_{\Sigma_t} \leq Q(n, q)
\]
Construction of the local canonical foliation.

1. *Lorentzian geodesic foliation* of the observer \((p, T_p)\).

   - \(\gamma : [0, \bar{c}r] \rightarrow M\): future-oriented, timelike geodesic with \(\gamma(\bar{c}r) = p\) and \(\dot{\gamma}(p) = T_p\). Set \(q = \gamma(0)\).

   - \(\bigcup \mathcal{H}_\tau\): a neigh. of \(p\) foliated by Lorentzian geodesic spheres centered at \(q\) in the past of \(p\)

   - \(y = (y^\alpha) = (\tau, y^i)\): normal coordinates associated with radial geodesics from the point \(q := \gamma(0)\).
2. Hessian comparison on the orthogonal complement

\[ E := (\nabla \tau)^\perp. \]

\[ k(\tau, r) g_{ij} \leq (-\nabla^2 \tau)|_{E,ij} \leq \bar{k}(\tau, r) g_{ij}, \]

with

\[ k(\tau, r) := \frac{r^{-1} \sqrt{C}}{\tan (\tau r^{-1} \sqrt{C})}, \quad \bar{k}(\tau, r) := \frac{r^{-1} \sqrt{C}}{\tanh (\tau r^{-1} \sqrt{C})}. \]

– Constant \( C \) depending only on the sup-norm of the curvature.
– Proof based on a Riccati-type argument, valid for timelike radial geodesics of a Lorentzian metric.

In particular, for the mean curvature

\[ n \underline{k}(\tau, r) \leq H_{\mathcal{H}_\tau} \leq n \bar{k}(\tau, r). \]

- Take \( p' = \gamma(\tau) \) with \( \tau \in [c_r, \bar{c}r] \).

For each \( a \in [c_r, \bar{c}r] \), consider the Riemannian slice \( \mathcal{A}(p', a) := S_{gT_q}(p', a) \cap \mathcal{J}^+(q) \) determined by the reference metric \( g_{T_q} \) associated with \( T_q \).

- Mean curvature estimate

\[
 n_k(a, r) \leq H_{\mathcal{A}(p', a)} \leq \overline{n_k}(a, r).
\]
4. Equations for the CMC foliation $\bigcup_t \Sigma_t$.

- Each hypersurface $\Sigma_t = \{(u^t(y), y)\}$ (with second fund. form $h_{ij}$) is a graph over a given geodesic slice $\mathcal{H}_\tau$ (with second fund. form $A_{ij}$)

- Mean curvature equation

$$\mathcal{M}u := h_{ij}g^{ij} = \frac{1}{\sqrt{1 + |\nabla \Sigma u|^2}} \left( \Delta \Sigma u + A^j_j \right), \quad A_{ij} := (\nabla^2_{\mathcal{M}\tau})_{ij}.$$ 

- Nonlinear elliptic PDE
  - The Lorentzian and Riemannian slices provide barriers.
  - Existence by the method of continuation, provided one establishes that $\Sigma$ is spacelike.
5. Localization of the CMC slices

- Fix $s \in [c, 2c]$ and introduce two points in the future of $p$, that is, $p_{sr} := \gamma(sr)$, $p'_{sr} := \gamma(s'r)$ with $s' = s + s^2$. Set $\Omega_{sr} \subset \mathcal{H}_{\tau=sr}$ whose boundary is

$$
\partial \Omega_{sr} := A(p'_s, s'sr) \cap \mathcal{H}_{\tau=sr}.
$$

- The hypersurface $\Sigma_t$ is the graph of the function $u$ given by the Dirichlet problem

$$
\mathcal{M}u = t \quad \text{in } \Omega_{sr},
$$

$$
u = sr \quad \text{in } \partial \Omega_{sr},
$$

with the following range for the mean curvature values

$$
t \in l(s, r) := [n\overline{k}(sr, r), n\underline{k}(2s^2r, r)].
$$
Sketch of the derivation of some of the uniform bounds.

– Quantitative estimates involving the curvature $R_m g$, only.
– Nash-Moser type technique

► Global gradient estimate ensuring that the prescribed mean curvature equation is uniformly elliptic. (See below.)

► Second fundamental form $h_{ij}$ controled via Simons identity

\[
\Delta h_{ij} = \Delta h_{ij} - (trh)_{ij} \\
= |h|^2 h_{ij} - (trh) h_{ik} h_{lj} g^{kl} - R_{ipjq} h_{kl} g^{pk} g^{ql} + R_{jplq} h_{ik} g^{pq} g^{kl} \\
+ \nabla_p (R_{qijN}) g^{pq} - \nabla_j (R_{iN}).
\]
Thanks to Weitzenböck’s identity, the following inequality is satisfied by the Laplacian of $|\nabla u|^2$ on the hypersurface $\Sigma_t$

$$\Delta |\nabla u|^2 - 2|\nabla^2 u|^2 \gtrsim \langle \nabla u, \nabla \Delta u \rangle - (1 + |\nabla u|^2)^3$$

with, moreover,

$$|\Delta u| \lesssim 1 + |\nabla u|^2 =: \nu (\nabla u)^2.$$
Lemma

The CMC hypersurfaces are spacelike:

\[ \| \nabla u \|_{L^\infty} = \sup_{\Omega_{sr}} |\nabla u| \lesssim 1. \]

Sketch of the proof.

**Step 1.** Estimate \( \| \nabla u \|_{L^\infty} \) in term of \( \| \nabla u \|_{L^{p_0}} \) for some finite \( p_0 \).

Set

\[ \nu = (\nu^2 - k)_+ := (1 + |\nabla u|^2 - k)_+, \]

with \( k \) so large that \( \nu = 0 \) on \( \partial\Sigma \).

Choosing such a \( k \) is possible, since the desired gradient estimate near the boundary follows from the maximum principle.
Given $q \geq 1$, multiply by $v^q$ our consequence of the Weitzenböck’s identity:

$$
\Delta |\nabla u|^2 - 2|\nabla^2 u|^2 \gtrsim \langle \nabla u, \nabla \Delta u \rangle - (1 + |\nabla u|^2)^3,
$$

integrate over $\Sigma$, and use also $|\Delta u| \lesssim 1 + |\nabla u|^2$.

We obtain

$$
\int_\Sigma \left( q v^{q-1} |\nabla v|^2 + v^q |\nabla^2 u|^2 \right) dv_\Sigma \\
\lesssim \int_\Sigma \left( q v^{q-1} \langle \nabla v, \nabla u \rangle + v^{q+3} + v^q \right) dv_\Sigma.
$$

Remove the term $|\nabla^2 u|^2$. 

Setting $q = 2m - 1$ we obtain for all $m \geq 1$

$$\|\nabla v^m\|_{L^2(\Sigma)}^2 \lesssim m^2 \|v^{2m+2} + v^{2m-2}\|_{L^1(\Sigma)}.$$ 

Rewritting this in coordinates $y^j$ in the geodesic slice and applying the Sobolev inequality (in a fixed compact domain)

$$\|w\|_{L^{2n/(n-1)}(\Omega_{sr})}^2 \lesssim \|g^{ij} \partial_i w \partial_j w + w^2\|_{L^1(\Omega_{sr})}$$

to the function $w := v^{p/2}$ with now $p := 2m - 1/2$, we deduce

$$\|v\|_{L^{pn/(n-1)}(\Omega_{sr})} \lesssim p^{2/p} \|v^{p+2} + v^{p-2}\|_{L^1(\Omega_{sr})}^{1/p}, \quad p > 2.$$
We control the $L^{pn/(n-1)}$ norm of $v$ in terms of its $L^p$ norm. Since $pn/(n - 1) < p$, an iteration procedure will allow us to control the sup norm of $v$.

Without loss of generality, assume that $\|v\|_{L^\infty(\Omega_{sr})} \geq 1$, for otherwise the result is immediate:

$$\max (1, \|v\|_{L^{pn/(n-1)}(\Omega_{rs})}) \lesssim p^{2/p} \|v\|^{2/p}_{L^\infty(\Omega_{rs})} \max (1, \|v\|_{L^p(\Omega_{rs})}),$$

and by iteration

$$\|v\|_{L^\infty(\Omega_{rs})} \lesssim \|v\|_{L^\infty(\Omega_{rs})}^{\alpha} \|v\|_{L^{p_0}(\Omega_{rs})} \|v\|_{L^{p_0}(\Omega_{rs})},$$

$$\alpha := \frac{2}{p_0} \sum_{k=0}^{\infty} (1 - 1/n)^k = \frac{2n}{p_0}.$$

It suffices to choose to take $p_0 > 2n$. 
Step 2. Uniform gradient estimate in a fixed $L^{p_0}$ norm.

From $|\Delta u| \lesssim |\nu(\nabla u)|^2$, we obtain for all $\lambda > 0$

$$\Delta(e^{\lambda u}) = \lambda^2 e^{\lambda u} |\nabla u|^2 + \lambda e^{\lambda u} \Delta u$$

$$\gtrsim \lambda^2 e^{\lambda u} |\nabla u|^2 - \lambda e^{\lambda u} |\nu(\nabla u)|^2.$$ 

From this and our “Weitzenböck’s inequality”, we deduce

$$\Delta \left( v^{p_0} e^{\lambda u} \right) \gtrsim - v^{p_0-1} e^{\lambda u} \left( \nu^2 (\nu^4 + \lambda \nu) - \lambda^2 (\nu - 1) \right)$$

$$+ \lambda p_0 v^{p_0-1} e^{\lambda u} \langle \nabla u, \nabla \nu \rangle$$

$$+ p_0 v^{p_0-1} e^{\lambda u} \langle \nabla u, \nabla (\Delta u) \rangle + p_0 (p_0 - 1) v^{p_0-2} e^{\lambda u} |\nabla \nu|^2.$$ 

Observe that $\nu^2 (\nu^4 + \lambda \nu) - \lambda^2 (\nu - 1) \lesssim \nu^3$, provided $k > 1$ is fixed and $\lambda$ is arbitrarily large.

Integrating over $\Sigma$, proceeding as in Step 1, and taking $\lambda$ to be arbitrary large, we arrive at

$$\int_{\Sigma} |\nabla u|^{p_0} d\nu_{\Sigma} \lesssim C_{p_0}.$$
4. Geometric estimates in past canonical neighborhoods

In collaboration with J. Grant (Vienna). Preprint ArXiv:1008.5167.

- Weaker curvature bounds
- Within causal sets
- Estimates proven in the same sets

Notation:

- \((M, g, p, T_p)\): time-oriented, pointed Lorentzian manifold.

- \(g_T\): positive definite canonical inner product on \(T_pM\)

\[
g_T(X, Y) := g(X, Y) + 2g(T_p, X)g(T_p, Y), \quad X, Y \in T_pM.
\]

- \(\langle X, Y \rangle_T := g_T(X, Y), \quad |X|_T^2 := g_T(X, X)\).
Definition

Given $\theta \in [0, \pi/2]$ and $r > 0$, introduce the cone $J_{r,\theta}^-(p, T_p) \subset J^-(p)$ of all past-oriented vectors $X \in T_p M$ satisfying

$$-r^2 \leq g(X, X) \leq 0, \quad \cos(\theta/2) \leq \frac{\langle X, T_p \rangle_T}{|X|_T} \leq 1.$$ 

Its image via the exponential map

$$J_{r,\theta}^-(p, T_p) := \exp_p \left( J_{r,\theta}^-(p, T_p) \right)$$

is referred to as a *past canonical neighborhood* of $(p, T_p)$. 
Sectional curvature bound.

- Standard definition

\[
K_g(X,Y) := \frac{\langle \text{Rm}(X,Y)Y,X \rangle}{g(X,X) g(Y,Y)}
\]

for a normalized timelike 2-plane \((X,Y)\)

\[
g(X,X) < 0, \quad g(X,Y) = 0, \quad g(Y,Y) > 0.
\]

- Past upper sectional constant of the observer \((p,T_p)\) within \(\mathcal{J}_{r,\theta}^-(p,T_p)\): best constant \(\Lambda = \text{Sec}_{r,\theta}^-(p,T_p)\) such that

\[
\langle \text{Rm}_g(X,Y)Y,X \rangle \leq \Lambda g(X,X) g(Y,Y),
\]

\[
X \in TG_{r,\theta}^-(p,T_p), \quad Y \in \{X\}^\perp,
\]

where \(TG_{r,\theta}^-(p,T_p)\) is the bundle of all tangents of radial geodesics from \(p\).
Ricci curvature bound.

- The past lower Ricci constant of an observer \((p, T_p)\) is defined as the best constant \(c = \mathcal{R}ic_{r,\theta}^{-}(p, T_p)\) in

\[
Rc_g(X, X) \geq n c g(X, X), \quad X \in TG_{r,\theta}^{-}(p, T_p).
\]
Past conjugate radius.

Definition

The *past conjugacy radius of the observer* \((p, T_p)\) is the supremum \(\text{Conj}_{\theta}^{-}(p, T_p)\) of all \(r\) such that

\[
\exp_p : J_{r,\theta}^{-}(p, T_p) \rightarrow J_{r,\theta}^{-}(p, T_p)
\]

is a local diffeomorphism.
Theorem (Lower bound on the past conjugacy radius of an observer).

\[
\text{Conj}^{-}_\theta (p, T_p) \geq \sup_{r > 0} \min \left( d^{-}_\theta (p, T_p; \partial M), \pi \Lambda^{-1/2} \right)
\]

\(d^{-}_\theta (p, T_p; \partial M)\) : distance from \(p\) to \(\partial M \setminus \{p\}\) within \(J^{-}_{\infty, \theta}(p, T_p)\)

\(\Lambda := \max (0, -\text{Sec}^{-}_{r, \theta}(p, T_p))\)

Equality achieved for manifolds isometrically embedded into the model spacetime of constant curvature.

**Remark.** Beem, Ehrlich, Kim, Harris.
Proof based on Jacobi field estimates.
Given a past-oriented, radial geodesic $\gamma$ from $p$, consider a Jacobi field $Y$ along $\gamma$:

$$
\ddot{Y}(s) + R_m g(Y(s), \dot{\gamma}(s)) \dot{\gamma}(s) = 0, \\
Y(0) = 0, \quad \dot{Y}(0) \neq 0,
$$

where $\dot{Y} := \nabla_{\dot{\gamma}} Y$.

For each $\Lambda \in \mathbb{R}$ we introduce

$$
\Phi_\Lambda(t) := \begin{cases} 
|\Lambda|^{-1/2} \sinh(|\Lambda|^{-1/2} t), & \Lambda > 0, \\
t, & \Lambda = 0, \\
|\Lambda|^{-1/2} \sin(|\Lambda|^{-1/2} t), & \Lambda < 0.
\end{cases}
$$

defined for all $t \geq 0$ if $\Lambda \geq 0$, but only for $t \in [0, \pi |\Lambda|^{-1/2}]$ if $\Lambda < 0$. 
Lemma

Let $\gamma : [0, 1] \to J_{\theta, r}^+(p, T_p)$ be a time-like radial $g$-geodesic. Let $Y$ be a Jacobi field along $\gamma$ with $Y(0) = 0$ and $Y(t) \perp \dot{\gamma}(t)$. Let $\Lambda \in \mathbb{R}$ be the upper Lorentzian sectional curvature constant:

$$K_g(\dot{\gamma}(t), X) \geq \Lambda, \quad X \in T_{\gamma(t)} M, \quad t \in [0, 1].$$

Then one has

$$|Y(t)| \geq |\dot{Y}(0)| \Phi_\Lambda(t), \quad t \in [0, 1].$$

In particular:

- If $\Lambda \geq 0$ then $\gamma(t)$ is never conjugate to $p$ along $\gamma$.
- If $\Lambda < 0$ then $\gamma(t)$ is not conjugate to $p$ along $\gamma$ provided $t < \pi / \sqrt{\lvert \Lambda \rvert}$. 
Past volume estimate.

\[ |\mathcal{J}_{r,\theta}(p, T_p)|: \text{Lorentzian volume} \]

\((\overline{M}_c, \overline{g}_c)\): simply-connected Lorentzian manifold of curvature \(c\)

\[ |\overline{\mathcal{J}}_{r,\theta}(c)|: \text{volume of canonical neighborhoods in } (\overline{M}_c, \overline{g}_c) \]
Theorem (Relative volume comparison estimate in past canonical neighborhoods).

Let \((M, g, p, T_p)\) be a pointed Lorentzian manifold with (smooth) boundary and let \(\theta \in [0, \pi/2)\).

Given \(r < \text{Conj}^{-\theta}(p, T_p)\), consider the model with constant curvature \(c =: \text{Ric}^{-\theta}_r(p, T_p)\). Then, the volume ratio

\[
\rho_{s,\theta}(p, T_p) := \frac{|J_{s,\theta}^{-}(p, T_p)|}{|\mathcal{J}_{s,\theta}(c)|}
\]

is non-increasing as a function of \(s \in [0, r]\).

Remark. Gromov, Ehrlich, Kim, Sanchez, etc.
Past injectivity radius

Let \((M, g)\) be a time-oriented, Lorentzian manifold with boundary. Let \((p, T_p)\) be an observer and \(\theta \in [0, \pi/2)\).

**Definition**

The *past injectivity radius of the observer* within the past cone \(\mathcal{J}_{\infty, \theta}^-(p, T_p)\),

\[
\text{Inj}_{\theta}^-(p, T_p),
\]

is the supremum of all values \(r\) for which

\[
\exp_p : J_{r, \theta}^-(p, T_p) \to J_{r, \theta}^-(p, T_p)
\]

is a global diffeomorphism.
In summary.

- “Bounded curvature” implies “controled geometry”
- Quantitative bounds
- CMC–harmonic radius of an observer

What is next?

- Convergence and compactness of sequences of spacetimes
- Boundary of the spacetimes / nature of singularities
- Penrose’s strong censorship conjecture

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